Expectation and Other Parameters

Expectation (denoted $E[X], \mu_X$, or μ) – For a random variable *X*, the expectation of *X* (aka expected value of *X,* or mean of *X*) is the weighted average of the values of *supp(X).* The weights are the corresponding values of the pdf.

For a discrete random variable we have

$$E[X] = \sum_{x \in \operatorname{supp}(X)} x \cdot p(x) = x_1 \cdot p(x_1) + x_2 \cdot p(x_2) + \cdots$$

For a continuous random variable we have

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)dx .$$

If *X* is non-negative, it can be shown that $E[X] = \int_0^{\infty} x \cdot f(x)dx = \int_0^{\infty} S(x)dx..$

If *h* is a function of the random variable *X*, then the expectation of *h(X)* is

i) If *X* is discrete, $E[h(X)] = \sum_{x \in \operatorname{supp}(X)} h(x) \cdot p(x)$

ii) If *X* is continuous, $E[h(X)] = \int_{-\infty}^{\infty} h(x) \cdot f(x)dx$.

Notice that the expectation formulas above are a special case of these formulas with *h(X) = X.*

Example: Suppose *X* is a discrete random variable with *p(0) = 0.5*, *p(1) = 0.2,* and *p(4) = 0.3*. Find $E[X^{0.5}]$, and separately find $E[3X + 2]$.

Let's again draw a probability distribution table.

X	$X^{0.5}$	$3X+2$	$p(x)$
0	0	2	0.5
1	1	5	0.2
4	2	14	0.3

Then $E[X^{0.5}] = E[\sqrt{X}] = 0 \cdot 0.5 + 1 \cdot 0.2 + 2 \cdot 0.3 = 0.8$ and $E[3X + 2] = 2 \cdot 0.5 + 5 \cdot 0.2 + 14 \cdot 0.3 = 6.2.$ Notice that since $E[X] = 0 + 0.2 + 1.2 = 1.4,$ then $E[X^{0.5}] \neq (E[X])^{0.5}$. However we do have that $E[3X + 2] = 3E[X] + 2.$ In general, we have the formula $E[a \cdot g(X) + b \cdot h(X) + c] = a \cdot E[g(X)] + b \cdot E[h(X)] + c$ where *a, b*, and *c* are constants.

Other Distribution Parameters and Relationships Among Them
The n^{th} moment of the random variable X is defined to be $E[X^n]$. If the mean of X is $\mu_X = \mu$, then the n^{th} central moment of X about the mean μ is $E[(X-\mu)^n]$.

The variance of the random variable X is denoted by $Var(X)$, $V(X)$, σ_X^2, or σ^2, and is defined to be the 2nd central moment of X about the mean μ.
We have

$$Var(X) = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] = E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - (E[X])^2$$

An important property of the variance is that if $Y = aX + b$, where a and b are constants, then

$$Var(Y) = Var(aX+b) = a^2 Var(X).$$

The standard deviation of the random variable X is the square root of the variance and is thus denoted by σ_X or σ. In symbols,

$$\sigma_X = \sqrt{Var(X)}$$

The coefficient of variation of the random variable X is the ratio of the standard deviation of X to the mean of X. In symbols,

$$CV_X = \frac{\sigma_X}{\mu_X}$$

The median of a distribution is the 50th percentile of the distribution.

The mode of a distribution is any value of the random variable X at which the pdf is maximized.

The moment generating function (mgf) of the random variable X is denoted $M_X(t)$, $m_X(t)$, $M(t)$, or $m(t)$ and is defined to be $M_X(t) = E[e^{tX}]$.

Properties of mgf's:

1. $M_X(0) = 1$

2. If X_1 and X_2 are random variables and $M_{X_1}(t) = M_{X_2}(t)$, then $X_1 \sim X_2$.

3. $E[X] = \frac{d}{dt} M_X(t)\,|_{t=0} = M'_X(0)$, and in general $E[X^n] = M_X^{(n)}(0)$.

4. If we define $R_X(t) = \ln(M_X(t))$, then

$$R'_X(0) = \frac{d}{dt} R_X(t)\,|_{t=0} = \frac{M'_X(t)}{M_X(t)}\,|_{t=0} = \frac{M'_X(0)}{M_X(0)} = \frac{E[X]}{1} = E[X], \text{ and similarly}$$

$$R''_X(0) = Var(X)$$

Chebyshev's Inequality: If X is a random variable with mean μ and standard deviation σ then for any real number $r > 0$ we have

$$\Pr(|X - \mu| > r \cdot \sigma) \leq \frac{1}{r^2}$$

A picture is worth a thousand words: